Solution 2 by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania. We denote: $x' = \frac{BA'}{A'C}$, $y' = \frac{CB'}{B'A}$, $z' = \frac{AC'}{C'B}$, $x'' = \frac{BA''}{A''C}$, $y'' = \frac{CB''}{B''A}$, $z'' = \frac{AC''}{C''B}$.
By Routh' theorem we have

$$[A'B'C'] = \frac{x'y'z'+1}{(x'+1)(y'+1)(z'+1)}[ABC], [A''B''C''] = \frac{x''y''z''+1}{(x''+1)(y''+1)(z''+1)}[ABC]$$

Because we have $x'y'z' = x''y''z'' = 1$, the inequality to prove becomes

$$\frac{27(x''+1)(y''+1)(z''+1)}{(x'+1)(y'+1)(z'+1)} \leq \left(\frac{x'(x''+1)}{x''(x'+1)} + \frac{y'(y''+1)}{y''(y'+1)} + \frac{z'(z''+1)}{z''(z'+1)}\right)^3$$

which yields immediately by AM-GM inequality.

Solution 3 by Michel Bataille, Rouen, France. Let P (resp. Q) be the point of concurrency of the cevians AA', BB', CC' (resp. AA'', BB'', CC''). In barycentric coordinates relatively to (A, B, C), we have $P = (x_1 : x_2 : x_3)$ and $Q = (y_1 : y_2 : y_3)$ where $x_1, x_2, x_3, y_1, y_2, y_3$ are positive real numbers and $x_1+x_2+x_3 = y_1+y_2+y_3 = 1$. With these notations, the coordinates of A', B', C' are

$$A' = (0 : x_2 : x_3), \quad B' = (x_1 : 0 : x_3), \quad C' = (x_1 : x_2 : 0) \tag{1}$$

and therefore $\frac{[A'B'C']}{[ABC]} = |\delta|$ where

$$\delta = \begin{vmatrix} 0 & \frac{x_1}{x_1+x_3} & \frac{x_1}{x_1+x_2} \\ \frac{x_2}{x_2+x_3} & 0 & \frac{x_2}{x_1+x_2} \\ \frac{x_3}{x_2+x_3} & \frac{x_3}{x_1+x_3} & 0 \end{vmatrix}.$$

We readily obtain $\frac{[A'B'C']}{[ABC]} = \frac{2x_1x_2x_3}{(x_1+x_2)(x_2+x_3)(x_1+x_3)}$; a similar result holds for $\frac{[A''B''C'']}{[ABC]}$ and it follows that the left-hand side of the inequality is $\mathcal{L}$ with

$$\mathcal{L} = \frac{27x_1x_2x_3(y_1+y_2)(y_2+y_3)(y_1+y_3)}{y_1y_2y_3(x_1+x_2)(x_2+x_3)(x_1+x_3)}.$$

From (1), we have $(x_2+x_3)A' = x_2B + x_3C$, hence $(x_2+x_3)\overrightarrow{BA'} = x_3\overrightarrow{BC}$ and so $BA' = \frac{x_3 \cdot BC}{x_2+x_3}$. Similarly, $BA'' = \frac{y_3 \cdot BC}{y_2+y_3}$ so that $\frac{BA'}{BA''} = \frac{x_3(y_2+y_3)}{y_3(x_2+x_3)}$. In the same way, we arrive at

$$\frac{CB'}{CB''} = \frac{x_1(y_1+y_3)}{y_1(x_1+x_3)}, \qquad \frac{AC'}{AC''} = \frac{x_2(y_1+y_2)}{y_2(x_1+x_2)}$$

and the right-hand side $\mathcal{R}$ writes as

$$\mathcal{R} = \left(\frac{x_3(y_2+y_3)}{y_3(x_2+x_3)} + \frac{x_1(y_1+y_3)}{y_1(x_1+x_3)} + \frac{x_2(y_1+y_2)}{y_2(x_1+x_2)}\right)^3.$$

The desired inequality $\mathcal{R} \geq \mathcal{L}$ now results from $(a_1+a_2+a_3)^3 \geq 27a_1a_2a_3$ (AM-GM) applied to

$$a_1 = \frac{x_3(y_2+y_3)}{y_3(x_2+x_3)}, \ a_2 = \frac{x_1(y_1+y_3)}{y_1(x_1+x_3)}, \ a_3 = \frac{x_2(y_1+y_2)}{y_2(x_1+x_2)}.$$

Also solved by the proposer.

63. *Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.* Let $a, b, c \in \mathbb{R}$. Prove that

$$9\sqrt{2}(ab(a-b) + bc(b-c) + ca(c-a)) \leq \sqrt{3}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)^{\frac{3}{2}}.$$

Solution 1 by Michel Bataille, Rouen, France. The inequality is obvious if $ab(a-b)+bc(b-c)+ca(c-a)\le 0$ and otherwise is equivalent to

$$54\left((a-b)(b-c)(a-c)\right)^2 \le \left((a-b)^2+(b-c)^2+(c-a)^2\right)^3 \qquad (1)$$

(since $ab(a-b)+bc(b-c)+ca(c-a)=(a-b)(b-c)(a-c)$). Let $\mathcal{L}(a,b,c)= 54\left((a-b)(b-c)(a-c)\right)^2$ and $\mathcal{R}(a,b,c)=\left((a-b)^2+(b-c)^2+(c-a)^2\right)^3$. If $a_1= a-c,\ b_1=b-c$ and $c_1=0$, then $a_1-b_1=a-b,\ b_1-c_1=b-c,\ a_1-c_1=a-c$ so that $\mathcal{L}(a_1,b_1,c_1)=\mathcal{L}(a,b,c)$ and $\mathcal{R}(a_1,b_1,c_1)=\mathcal{R}(a,b,c)$. It follows that it suffices to prove (1) in the case when $c=0$, that is, to show that $54(a-b)^2a^2b^2\le \left((a-b)^2+b^2+a^2\right)^3$ or equivalently,

$$27a^2b^2(a-b)^2\le 4(a^2+b^2-ab)^3. \qquad (2)$$

Now, it is straightforward to check the identity

$$4(a^2+b^2-ab)^3-27a^2b^2(a-b)^2=(a-2b)^2(2a-b)^2(a+b)^2$$

so that (2) writes as $(a-2b)^2(2a-b)^2(a+b)^2\ge 0$ and clearly holds.

Solution 2 by Arkady Alt, San Jose, California, USA. Due to cyclic symmetry of inequality we may assume that $a=\max\{a,b,c\}$. Since the inequality is obviously holds if $b<c$ (because then
$ab(a-b)+bc(b-c)+ca(c-a)=(a-b)(a-c)(b-c)\le 0$) suffice to consider only case when $b\ge c$, that is $a\ge b\ge c$. Let $x=b-c, y=a-b, p=x+y, q=xy$. Then $x,y\ge 0, a=c+x+y, b=c+x$,
$ab(a-b)+bc(b-c)+ca(c-a)=(x+y)\,xy=pq, (a-b)^2+(b-c)^2+(c-a)^2= \left(x^2+y^2+(x+y)^2\right)=2\left(x^2+y^2+xy\right)=2\left(p^2-q\right)$ and in the new notation the inequality is

$9\sqrt{2}pq\le\sqrt{3}\left(2\left(p^2-q\right)\right)^{3/2}$, where $q\ge 0$ and $q\le\dfrac{p^2}{4}$ (condition of solvability of Vieta's System $\begin{cases} x+y=p\\ xy=q\end{cases}$ in real x,y). We have $\sqrt{3}\left(2\left(p^2-q\right)\right)^{3/2}-9\sqrt{2}pq\ge$ $\sqrt{3}\left(2\left(p^2-\dfrac{p^2}{4}\right)\right)^{3/2}-9\sqrt{2}p\cdot\dfrac{p^2}{4}=\sqrt{3}\left(\dfrac{3p^2}{2}\right)^{3/2}-\dfrac{9p^3}{2\sqrt{2}}=\dfrac{9p^3}{2\sqrt{2}}-\dfrac{9p^3}{2\sqrt{2}}=0.$

Also solved by Kevin Soto Palacios, Huarmey,Peru; Ravi Prakash, New Delhi, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania and the proposer.

64. *Problem proposed by Arkady Alt, San Jose, California, USA.* Let $\Delta(x,y,z):= 2(xy+yz+xz)-(x^2+y^2+z^2)$ and let a,b,c be sidelengths of a triangle with area F. Prove that

$$\Delta\left(a^3,b^3,c^3\right)\le\frac{64F^3}{\sqrt{3}}.$$

Solution by Michel Bataille, Rouen, France. In the featured solution of problem 1973 in *Mathematics Magazine*, Vol. 89, No 4, October 2016, p. 297, it is proved that

$$\Delta(a,b,c)\cdot\Delta(a^3,b^3,c^3)\le\left(\Delta(a^2,b^2,c^2)\right)^2 \qquad (1)$$

whenever a,b,c are positive real numbers. Taking for a,b,c the sidelengths of the triangle, we calculate

$$\Delta(a,b,c)=2(ab+bc+ca)-(a^2+b^2+c^2)=2(s^2+r^2+4rR)-(2s^2-2r^2-8rR)=4r(r+4R)>0$$